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Fuzzy Proximity Spaces

A. K. KATSARAS

Department of Mathematics, University of Ioannina, Ioannina, Greece

Submitted by L. Zadeh

1. INTRODUCTION

The concept of a fuzzy set was introduced by Zadeh in [13] and was later used by many other authors in various branches of Mathematics. Chang [1] used the idea of a fuzzy set to introduce fuzzy topological spaces. Several other authors continued the investigation of such spaces. The concept of a fuzzy uniform space was given by Lowen in [6]. In this paper we will use fuzzy sets to define fuzzy proximity spaces and study some of their properties. To each such space corresponds a fuzzy topological space.

2. PRELIMINARIES

In this section we will recall some of the definitions related to fuzzy sets, fuzzy topological spaces and fuzzy uniform spaces.

Let X be a set and I the unit interval. A fuzzy set in X is an element of the set I^X of all functions μ from X into I. If f is a function from X into Y and $\mu \in I^Y$, then $f^{-1}(\mu)$ is the element of I^X which is defined by $f^{-1}(\mu)(x) = \mu(f(x))$. Also, for $\sigma \in I^X$, $f(\sigma)$ is the member of I^Y defined by

$$f(\sigma)(y) = \begin{cases} \sup_{x \in f^{-1}[y]} \sigma(x) & \text{if } f^{-1}[y] \text{ is not empty} \\ = 0 & \text{otherwise.} \end{cases}$$

A fuzzy topology on X is a subset α of I^X such that

(i) $0, 1 \in \alpha$

(ii) If $\mu, \rho \in \alpha$, then $\mu \wedge \rho \in \alpha$

(iii) If $\mu_i \in \alpha$ for each $i \in A$, then $\sup_{i \in A} \mu_i \in \alpha$.

A closure operator an I^X is a map $\mu \to \bar{\mu}$ from I^X into I^X such that for all μ, ρ in I^X we have

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(1) $\mu \leqslant \bar{\mu}$

(2) $\bar{\bar{\mu}} = \mu$

(3) $\overline{\mu \vee \rho} = \bar{\mu} \vee \bar{\rho}$

(4) $\bar{0} = 0$.

Given a closure operator on I^X, the collection

$$\{\mu: \overline{1 - \mu} = 1 - \mu\}$$

defines a fuzzy topology on X.

Let $\rho \in I^X$ and $\mu \in I^{X \times X}$. We define $\mu\langle\rho\rangle \in I^X$ (see [7]) by

$$\mu\langle\rho\rangle(x) = \sup_{y \in X} \rho(y) \wedge \mu(y, x).$$

For $\mu, \nu \in I^{X \times X}$, the composition $\mu \Delta \nu$ is defined by

$$\mu \Delta \nu(x, y) = \sup_{z \in X} \nu(x, z) \wedge \mu(z, y).$$

A fuzzy uniformity on X is a subset $\mathscr{U}$ of $I^{X \times X}$ such that

(1) $\mu, \nu \in \mathscr{U}$ implies $\mu \wedge \nu \in \mathscr{U}$

(2) $\mu \geqslant \nu \in \mathscr{U}$ implies $\mu \in \mathscr{U}$

(3) For every $\mu \in \mathscr{U}$ and every $x \in X$ we have $\mu(x, x) = 1$

(4) For every $\mu \in \mathscr{U}$, we have $\tilde{\mu} \in \mathscr{U}$ where $\tilde{\mu}(x, y) = \mu(y, x)$.

(5) Given $\mu \in \mathscr{U}$ there exists $\nu \in \mathscr{U}$ with $\nu \Delta \nu \leqslant \mu$.

A fuzzy uniformity $\mathscr{U}$ on X defines a fuzzy topology $\tau(\mathscr{U})$ by

$$\tau(\mathscr{U}) = \{\mu \in I^X: \psi(1 - \mu) = 1 - \mu\}$$

where, for $\mu \in I^X$, $\psi(\mu) = \inf_{\alpha \in \mathscr{U}} \alpha\langle\mu\rangle$. An element α of $\mathscr{U}$ is called symmetric if $\tilde{\alpha} = \alpha$. It is clear that, for each $\alpha \in \mathscr{U}$, the $\alpha \wedge \tilde{\alpha}$ is symmetric.

Let (X, τ_1) and (Y, τ_2) be two fuzzy topological spaces. A map $f: X \to Y$ is called continuous if $f^{-1}(\mu) \in \tau_1$ for each $\mu \in \tau_2$. If $\mathscr{U}$ and $\mathscr{F}$ are fuzzy uniformities on X and Y respectively, then a function f from X into Y is called uniformly continuous (with respect to $\mathscr{U}$ and $\mathscr{F}$) if, for each $\alpha \in \mathscr{F}$, the function $(x, y) \mapsto \alpha(f(x), f(y))$ is a member of $\mathscr{U}$.

The definition of the product of a family of fuzzy topological spaces is as defined in [12]. Finally, if A is a subset of X, the characteristic function of A will be denoted by χ_A.

3. Fuzzy Proximity Spaces

Recall that a binary relation δ on the power set of a set X is called a proximity on X if δ satisfies the following axioms (see, for example, [8], page 7):

(p_1) $A\delta B$ implies $B\delta A$

(p_2) $(A \cup B)\,\delta C$ iff $A\delta C$ or $B\delta C$

(p_3) $A\delta B$ implies $A \neq \varnothing$, $B \neq \varnothing$

(p_4) $A\bar{\delta}B$ implies that there exists a subset E of X such that $A\bar{\delta}E$ and $(X - E)\,\bar{\delta}B$.

(p_5) $A \cap B \neq \varnothing$ implies $A\delta B$.

Generalizing the notion in the case of fuzzy sets, we give the following definition of a fuzzy proximity space.

Definition 3.1. A binary relation δ on I^X is called a fuzzy proximity if δ satisfies the following axioms:

(FP1) $\mu\delta\rho$ implies $\rho\delta\mu$

(FP2) $(\mu \vee \rho)\,\delta\sigma$ iff $\mu\delta\sigma$ or $\rho\delta\sigma$

(FP3) $\mu\delta\rho$ implies $\mu \neq 0$ and $\rho \neq 0$

(FR4) $\mu\bar{\delta}\rho$ implies that there exists a $\sigma \in I^X$ such that $\mu\bar{\delta}\sigma$ and $(1 - \sigma)\,\bar{\delta}\rho$

(FP5) $\mu \wedge \rho \neq 0$ implies $\mu\delta\rho$.

The pair (X, δ) is called a fuzzy procimity space.

We get easily the following Lemma.

(3.2) Lemma. *Let (X, δ) be a fuzzy proximity space. Then:*

(1) *If $\mu\delta\rho$, $\mu_1 \geqslant \mu$, $\rho_1 \geqslant \rho$, then $\mu_1\delta\rho_1$*

(2) *$\mu\delta\mu$ for each $\mu \neq 0$*

(3) *$\mu\delta 1$ iff $\mu \neq 0$.*

Let now δ be a fuzzy proximity on X. For $\mu \in I^X$, we define

$$\bar{\mu} = 1 - \sup\{\rho \in I^X : \mu\bar{\delta}\rho\}.$$

Then, we have the following

(3.3) Theorem. *Let (X, δ) be a fuzzy proximity space. Then the map $\mu \to \bar{\mu}$ is a closure operator on I^X. Therefore, the collection*

$$\tau(\delta) = \{\mu \in I^X : \overline{1 - \mu} = 1 - \mu\}$$

is a fuzzy topology on X.

Proof. (a) $\mu \leqslant \bar{\mu}$.

In fact, let $x \in X$. If $\rho \bar{\delta} \mu$, then $\rho \wedge \mu = 0$ and hence either $\mu(x) = 0$ or $\rho(x) = 0$. In both cases we have $\rho(x) \leqslant 1 - \mu(x)$. Thus

$$\sup_{\rho \bar{\delta} \mu} \rho(x) \leqslant 1 - \mu(x)$$

which shows that $\bar{\mu}(x) \geqslant \mu(x)$.

(b) $\bar{\bar{\mu}} = \bar{\mu}$.

For this, it suffices to show that $\rho \delta \mu$ iff $\rho \delta \bar{\mu}$. Suppose, by way of contradiction, that $\rho \delta \bar{\mu}$ but $\rho \bar{\delta} \mu$. Then there exists $\gamma \in I^X$ such that $\gamma \bar{\delta} \rho$, $(1 - \gamma) \bar{\delta} \mu$. Since $\rho \delta \bar{\mu}$ and $\rho \bar{\delta} \gamma$, there exists an $x \in X$ such that $\gamma(x) < \bar{\mu}(x)$. Choose $\gamma(x) < a < \bar{\mu}(x)$ and let $\rho \in I^X$ be defined by

$$\begin{aligned} \rho_x(y) &= 1 - a, \qquad & y &= x \\ &= 0, & y &\neq x. \end{aligned}$$

Then $1 - \gamma \geqslant \rho_x$. Also $\rho_x \delta \mu$ because otherwise we would have $\bar{\mu}(x) \leqslant 1 - \rho_x(x) = a$. Thus, from the $\rho_x \delta \mu$ and $1 - \gamma \geqslant \rho_x$, we get $(1 - \gamma) \delta \mu$, a contradiction.

(c) $\overline{\mu \vee \rho} = \bar{\mu} \vee \bar{\rho}$.

It is easy to see that $\overline{\mu \vee \rho} \geqslant \bar{\mu} \vee \bar{\rho}$. On the hand, suppose that there exists an $x \in X$ and $\epsilon > 0$ with

$$a = \overline{\mu \vee \rho}(x) > [\bar{\mu}(x) \vee \bar{\rho}(x)] + \epsilon.$$

Suppose that $\bar{\rho}(x) \leqslant \bar{\mu}(x)$ (the case $\bar{\mu}(x) < \bar{\rho}(x)$ is similar). Since $\bar{\mu}(x) < a - \epsilon$, there exists $\gamma \in I^X$ such that $\gamma \bar{\delta} \mu$ and $1 - \gamma(x) < a - \epsilon$. Also, from the $1 - \gamma(x) \geqslant \bar{\mu}(x) \geqslant \bar{\rho}(x) > \bar{\rho}(x) - \epsilon/2$ follows that there exists $\lambda \in I^X$, $\lambda \bar{\delta} \rho$ and $1 - \lambda(x) < 1 - \gamma(x) + \epsilon/2$ and thus $\lambda(x) > \gamma(x) - \epsilon/2$. Now, $(\lambda \wedge \gamma) \bar{\delta} (\mu \vee \rho)$ and $\lambda \wedge \gamma(x) > \gamma(x) - \epsilon/2$ which implies that

$$a = \overline{\mu \vee \rho}(x) \leqslant 1 - \lambda \wedge \gamma(x) < 1 - \gamma(x) + \epsilon/2 < a - \epsilon + \epsilon/2 = a - \epsilon/2$$

a contradiction. This contradiction proves (c). It is also easy to see that $\bar{0} = 0$. This shows that the map $\mu \to \bar{\mu}$ is a closure operator on I^X and therefore $\tau(\delta)$ is a fuzzy topology on X.

(3.4) Definition. Let (X, δ_1) and (Y, δ_2) be two fuzzy proximity spaces. A function $f: X \to Y$ is said to be a proximity (or a proximally) continuous) mapping iff, for all μ, ρ in I^X, $\mu \delta_1 \rho$ implies $f(\mu)\, \delta_2 f(\rho)$. If $X = Y$ and the identity map is proximally continuous, then we say that δ_1 is finer than δ_2 and that δ_2 is coarser than δ_1.

(3.5) THEOREM. *Let (X, δ_1), (Y, δ_2) be two fuzzy proximity spaces and $f: X \to Y$ a proximity mapping. Then f is continuous with respect to the corresponding fuzzy topologies $\tau(\delta_1)$ and $\tau(\delta_2)$.*

Proof. Let $\mu \in \tau(\delta_2)$. Since f is a proximity mapping, $\rho\bar{\delta}(1-\mu)$ implies that $f^{-1}(\rho)\,\bar{\delta}(1 - f^{-1}(\mu))$. Thus, for $\rho\bar{\delta}(1-\mu)$, we have

$$[1 - f^{-1}(\rho)]\,(x) \geqslant \overline{1 - f^{-1}(\mu)}\,(x)$$
$$1 - \rho(f(x)) \geqslant \overline{1 - f^{-1}(\mu)}\,(x).$$

Hence,

$$\overline{1 - f^{-1}(\mu)}\,(x) \leqslant \inf_{\rho\bar{\delta}(1-\mu)} (1-\rho)\,(f(x)) = \overline{1-\mu}\,(f(x) = (1-\mu)\,(f(x))$$
$$= 1 - f^{-1}(\mu)\,(x).$$

This proves that $f^{-1}(\mu) \in \tau(\delta_1)$ and the Theorem is proved.

The following Theorem gives an alternate description of a fuzzy proximity.

(3.6) THEOREM. *Let δ be a fuzzy proximity on X and let Σ denote the set of all functions $\varphi: I^X \to I^X$ such that for each $\mu \in I^X$ we have $\mu\bar{\delta}(1 - \varphi(\mu))$. Then Σ has the following properties:*

(1) *If $\varphi \in \Sigma$ and $\mu \in I^X$, then $\mu \leqslant \varphi(\mu)$.*

(2) *If φ_1, φ_2 are members of Σ, then $\varphi = \varphi_1 \wedge \varphi_2$ belongs to Σ, where φ is defined by $\varphi(\mu) = \varphi_1(\mu) \wedge \varphi_2(\mu)$.*

(3) *If φ is a mapping from I^X into I^X and if for each $\mu \in I^X$ there exists $\varphi_1 \in \Sigma$ such that $\varphi_1(\mu) \leqslant \varphi(\mu)$, then $\varphi \in \Sigma$.*

(4) *If $\varphi \in \Sigma$ and $\mu, \rho \in I^X$ are such that $\varphi(\mu) \wedge \rho = 0$, then there exists $\varphi_1 \in \Sigma$ with $\varphi_1(\mu) \wedge \varphi_1(\rho) = 0$.*

(5) *If $\varphi \in \Sigma$ and $\mu, \rho \in I^X$ are such that $\varphi(\mu) \wedge \rho = 0$, then there $\varphi_1 \in \Sigma$ with $\varphi_1(\rho) \wedge \mu = 0$.*

(6) *$\mu\delta\rho$ iff for each $\varphi \in \Sigma$ we have $\varphi(\mu) \wedge \rho \neq 0$.*

Conversely, if Σ is a family of functions from I^X to I^X satisfying (1)–(5), *then the binary relation δ on I^X, which is defined by $\mu\delta\rho$ iff $\varphi(\mu) \wedge \rho \neq 0$ for each $\varphi \in \Sigma$, is a fuzzy proximity on X.*

Proof. (1) Let $\varphi \in \Sigma$, $\mu \in I^X$ and $x \in X$. Since $\mu\bar{\delta}(1 - \varphi(\mu))$, we have $\mu \wedge (1 - \varphi(\mu)) = 0$. Thus $\mu(x) \neq 0$ implies $\varphi(\mu)\,(x) = 1$ and hence $\mu(x) \leqslant \varphi(\mu)\,(x)$.

(2) Let $\varphi_1, \varphi_2 \in \Sigma$, $\mu \in I^X$ and $\varphi = \varphi_1 \wedge \varphi_2$. Since $1 - \varphi(\mu) = [1 - \varphi_1(\mu)] \vee [1 - \varphi_2(\mu)]$ and since $\mu\bar{\delta}(1 - \varphi_i(\mu))$, $i = 1, 2$, we have $\mu\bar{\delta}(1 - \varphi(\mu))$ and that proves that $\varphi \in \Sigma$.

(3) Let φ satisfy the hypothesis of (3) and let $\mu \in I^X$. By hypothesis, there exist $\varphi_1 \in \Sigma$ with $\varphi(\mu) \geqslant \varphi_1(\mu)$. Since $\mu \bar{\delta}(1 - \varphi_1(\mu))$ and since $1 - \varphi(\mu) \leqslant 1 - \varphi_1(\mu)$, it follows that $\mu \bar{\delta}(1 - \varphi(\mu))$ which proves that $\varphi \in \Sigma$.

(4) Let $\varphi \in \Sigma$ and suppose that $\varphi(\mu) \wedge \rho = 0$.

Case I. $\mu = \rho$. From the $\varphi(\mu) \wedge \rho = 0$ and from (1) follows that $\mu = \rho = 0$. Define $\varphi_1 \colon I^X \to I^X$ by $\varphi_1(\mu) = 0$ and $\varphi_1(\sigma) = 1$ if $\mu \neq \sigma$. Then $\varphi_1 \in \Sigma$ and $\varphi_1(\mu) \wedge \varphi_1(\rho) = 0$.

Case II. $\mu \neq \rho$. Since $\rho \leqslant 1 - \varphi(\mu)$ and since $\mu \bar{\delta}(1 - \varphi(\mu))$, we have $\mu \bar{\delta} \rho$. There exists $\gamma_1 \in I^X$ with $\mu \bar{\delta} \gamma_1$, $\rho \bar{\delta}(1 - \gamma_1)$. Also, there exists $\gamma_2 \in I^X$ such that $\gamma_1 \bar{\delta} \gamma_2$ and $\mu \bar{\delta}(1 - \gamma_2)$. Define $\varphi_1(\mu) = \gamma_2$, $\varphi_1(\rho) = \gamma_1$ and $\varphi_1(\sigma) = 1$ for each $\sigma \in I^X$ different from μ and ρ. Then $\varphi_1 \in \Sigma$ and $\varphi_1(\mu) \wedge \varphi_1(\rho) = \gamma_1 \wedge \gamma_2 = 0$ since $\gamma_1 \bar{\delta} \gamma_2$.

(5) It follows from (4) since $\varphi_1(\mu) \geqslant \mu$ for each $\varphi_1 \in \Sigma$.

(6) Suppose that $\mu \bar{\delta} \rho$. There exists $\gamma \in I^X$ such that $\mu \bar{\delta}(1 - \gamma)$, $\rho \bar{\delta} \gamma$. Define $\varphi \colon I^X \to I^X$ by $\varphi(\mu) = \gamma$ and $\varphi(\sigma) = 1$ if $\sigma \neq \mu$. Then $\varphi \in \Sigma$ and $\varphi(\mu) \wedge \rho = 0$ since $\gamma \bar{\delta} \rho$. Conversely, suppose that for some $\varphi \in \Sigma$ we have $\varphi(\mu) \wedge \rho = 0$. Then $\rho \leqslant 1 - \varphi(\mu)$. Thus $\mu \bar{\delta} \rho$ since $\mu \bar{\delta}(1 - \varphi(\mu))$.

Conversely, suppose that Σ is a family of functions from I^X into I^X satisfying (1)–(5) and define a binary relation δ on I^X by $\mu \delta \rho$ iff $\varphi(\mu) \wedge \rho \neq 0$ for each φ in Σ. We will show that δ is a fuzzy proximity on X. The (FP1), (FP3) and (FP5) follow directly from the definition of δ and from the (1) and (5). Also, it is easy to see that if $\mu \delta \sigma$ or $\rho \delta \sigma$, then $(\mu \vee \rho)\, \delta \sigma$. Conversely, suppose that $\mu \bar{\delta} \sigma$ and $\rho \bar{\delta} \sigma$. There are φ_1, $\varphi_2 \in \Sigma$ such that $\varphi_1(\sigma) \wedge \mu = 0$, $\varphi_1(\sigma) \wedge \rho = 0$. Let $\varphi = \varphi_1 \wedge \varphi_2$. Then $\varphi \in \Sigma$ and $\varphi(\sigma) \wedge (\mu \vee \rho) = 0$ which implies that $(\mu \vee \rho)\, \bar{\delta} \sigma$. This proves (FP2). Finally, to prove (FP4), assume that $\mu \bar{\delta} \rho$. By (4), there exists $\varphi \in \Sigma$ such that $\varphi(\mu) \wedge \varphi(\rho) = 0$. Hence the sets

$$A = \{x \colon \varphi(\mu)\,(x) \neq 0\}, \qquad B = \{x \colon \varphi(\rho)\,(x) \neq 0\}$$

are disjoint. Let $\gamma = \chi_B$. Then $\gamma = 0$ on A and hence $\varphi(\mu) \wedge \gamma = 0$ which implies that $\mu \bar{\delta} \gamma$. Also $\varphi(\rho) \wedge (1 - \gamma) = 0$ and hence $\rho \bar{\delta}(1 - \gamma)$. This completes the proof.

We use now the preceding Theorem to show that a fuzzy uniformity on X induces a fuzzy proximity.

(3.7) THEOREM. *Let $\mathscr{U}$ be a fuzzy uniformity on X and let $\Sigma = \Sigma_{\mathscr{U}}$ be the family of all functions $\varphi \colon I^X \to I^X$ such that for each $\mu \in I^X$ there exists $\alpha \in \mathscr{U}$ with $\alpha\langle\mu\rangle \leqslant \varphi(\mu)$. Then Σ has the properties* (1)–(5) *of the preceding Theorem. Thus $\mathscr{U}$ induces a fuzzy proximity $\delta = \delta_{\mathscr{U}}$ on X defined by $\mu \delta \rho$ iff $\alpha\langle\mu\rangle \wedge \rho \neq 0$ for all $\alpha \in \mathscr{U}$.*

Proof. (1) and (3) are obvious since $\mu \leqslant \alpha\langle\mu\rangle$ for each $\alpha \in \mathscr{U}$. To prove (2), let $\varphi = \varphi_1 \wedge \varphi_2$ where $\varphi_1, \varphi_2 \in \Sigma$. Given $\mu \in I^X$, choose α_1, α_2 in $\mathscr{U}$ such that $\alpha_1\langle\mu\rangle \leqslant \varphi_1(\mu)$ and $\alpha_2\langle\mu\rangle \leqslant \varphi_2(\mu)$. Let $\alpha = \alpha_1 \wedge \alpha_2$. Then $\alpha \in \mathscr{U}$ and

$$\alpha\langle\mu\rangle \leqslant \alpha_1\langle\mu\rangle \wedge \alpha_2\langle\mu\rangle \leqslant \varphi_1(\mu) \wedge \varphi_2(\mu) = \varphi(\mu).$$

This shows that $\varphi \in \Sigma$. Since (5) follows from (4) and (1), it only remains to prove (4). So, let $\varphi \in \Sigma$ and μ, $\rho \in I^X$ with $\varphi(\mu) \wedge \rho = 0$. Choose $\alpha \in \mathscr{U}$ with $\alpha\langle\mu\rangle \leqslant \varphi(\mu)$ and a symmetric $\alpha_1 \in \mathscr{U}$ with $\alpha_1 \Delta \alpha_1 \leqslant \alpha$. We will show that $\alpha_1\langle\mu\rangle \wedge \alpha_1\langle\rho\rangle = 0$ which will finish the proof since $\alpha_1 \in \Sigma$. Suppose, by way of contradiction, that, for some x in X, we have $\alpha_1\langle\mu\rangle\,(x) \neq 0$ and $\alpha_1\langle\rho\rangle\,(x) \neq 0$. There are y, z in X such that $\mu(y) \neq 0$, $\alpha_1(y, x) \neq 0$, $\rho(z) \neq 0$, $\alpha_1(z, x) = \alpha_1(x, z) \neq 0$. Thus

$$\alpha(y, z) \geqslant \alpha_1 \Delta \alpha_1(y, z) \neq 0$$

and therefore $\alpha\langle\mu\rangle\,(z) \neq 0$ and hence $\varphi(\mu) \wedge \rho \geqslant \alpha\langle\mu\rangle \wedge \rho \neq 0$ a contradiction. This completes the proof.

(3.8) *Remark.* If $\mathscr{U}$ is a fuzzy uniformity, then $\delta = \delta_{\mathscr{U}}$ can equivalently be defined by

$\mu\delta\rho$ iff $\alpha\langle\mu\rangle \wedge \alpha\langle\rho\rangle \neq 0$ for each $\alpha \in \mathscr{U}$, or

$\mu\delta\rho$ iff $\alpha \wedge (\mu \times \rho) \neq 0$ for each $\alpha \in \mathscr{U}$ where $\mu \times \rho$ is defined on $X \times X$ by $\mu \times \rho(x, y) = \min\{\mu(x), \rho(y)\}$.

(3.9) Theorem. *Let $(X, \mathscr{U})$ and $(Y, \mathscr{F})$ be fuzzy uniform spaces and $f: X \to Y$ a uniformly continuous function. Then f is proximally continuous with respect to the induced fuzzy proximities $\delta_{\mathscr{U}}$ and $\delta_{\mathscr{F}}$.*

Proof. Let μ, ρ in I^X be such that $\mu\delta_{\mathscr{U}}\rho$ and let $\alpha \in \mathscr{F}$. Since f is uniformly continuous, the map $\alpha_1(x, y) = \alpha(f(x), f(y))$ is an element of $\mathscr{U}$. Our hypothesis implies that $\alpha_1\langle\mu\rangle \wedge \rho \neq 0$ which in turn implies that $\alpha\langle f(\mu)\rangle \wedge f(\rho) \neq 0$. This proves that $f(\mu)\,\delta_{\mathscr{F}} f(\rho)$ and the proof is complete.

4. Initial Fuzzy Proximities

Let $(X_\alpha, \delta_\alpha)_{\alpha \in A}$ be a family of fuzzy proximity spaces, X a set and for each $\alpha \in A$ let $f_\alpha: X \to X_\alpha$ be a map. For μ, $\rho \in I^X$, we define $\mu\delta\rho$ iff the following condition is satisfied: If $\mu = \mu_1 \vee \cdots \vee \mu_n$ and $\rho = \rho_1 \vee \cdots \vee \rho_k$ $(\mu_i, \rho_j \in I^X)$, then there exist i, j such that for each $\alpha \in A$ we have $f_\alpha(\mu_i)\,\delta_\alpha f_\alpha(\rho_j)$. As the following Theorem shows, δ is a fuzzy proximity on X and coincides with the coarsest proximity on X for which each f_α is proximally continuous.

(4.1) Theorem. (a) *The relation δ defined above is the coarsest proximity on X for which each f_α is a proximity mapping.*

(b) *Let* (Y, δ_1) *be a fuzzy proximity space and let* f *be a function from* Y *to* X. *Then* f *is a proximity mapping iff each* $h_\alpha = f_\alpha \circ f\colon Y \to X_\alpha$ *is proximally continuous.*

Proof. (a) It is easy to see that $\mu\delta\rho$ implies $\rho\delta\mu$ and $\mu\delta\sigma$ for each $\sigma \geqslant \rho$. Hence $\mu\delta\rho$ or $\mu\delta\sigma$ implies $\mu\delta(\rho \vee \sigma)$. On the other hand, suppose that $\mu\bar{\delta}\rho$ and $\mu\bar{\delta}\sigma$. There are $\mu_1, \ldots, \mu_n$, $\mu_1', \ldots, \mu_k'$, $\rho_1, \ldots, \rho_m$, $\sigma_1, \ldots, \sigma_l$ such that

$$\mu = \mu_1 \vee \cdots \vee \mu_n = \mu_1' \vee \cdots \vee \mu_k',$$

$$\rho = \rho_1 \vee \cdots \vee \rho_m, \qquad \sigma = \sigma_1 \vee \cdots \vee \sigma_l$$

and for each pair (i, j), $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, (resp. $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant l$) there exists an α with $f_\alpha(\mu_i)\, \bar{\delta}_\alpha f_\alpha(\rho_i)$ (resp. $f_\alpha(\mu_i')\, \bar{\delta}_\alpha f_\alpha(\sigma_j)$). It is clear that

$$\mu = \sup_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant k}} \mu_i \wedge \mu_j' \qquad \text{and} \qquad \rho \vee \sigma = \rho_1 \vee \cdots \vee \rho_m \vee \sigma_1 \vee \cdots \vee \sigma_l\,.$$

Let now $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant k$, $1 \leqslant r \leqslant m + l$. If $1 \leqslant r \leqslant m$, then there exists $\alpha \in A$ with $f_\alpha(\mu_i)\, \bar{\delta}_\alpha(\rho_r)$ and hence $f_\alpha(\mu_i \wedge \mu_j')\, \bar{\delta}_\alpha(\rho_r)$. Similarly, if $m + 1 \leqslant r \leqslant l + m$, there exists $\alpha \in A$ such that $f_\alpha(\mu_i \wedge u_j')\, \bar{\delta}_\alpha f_\alpha(\sigma_{r-m})$. This proves that $\mu\bar{\delta}(\rho \vee \sigma)$.

Since it is easily verified that (FP3) and (FP5) hold, it only remains to show (FP4). To this end, consider the set V of all pairs (μ, ρ) such that $\mu\bar{\delta}\rho$ and for every $\gamma \in I^X$ we have either $\mu\delta\gamma$ or $\rho\delta(1 - \gamma)$. We claim that V is empty. Assume the contrary. We first show that if $(\mu, \rho) \in V$, then $f_\alpha(\mu)\, \delta_\alpha f_\alpha(\rho)$ for each $\alpha \in A$. In fact, let $\gamma \in I^{X_\alpha}$ and $\lambda = f_\alpha^{-1}(\gamma)$. If $\mu\delta\lambda$, then $f_\alpha(\mu)\, \delta_\alpha f_\alpha(\lambda)$ and hence $f_\alpha(\mu)\, \delta_\alpha \gamma$ since $\gamma \geqslant f_\alpha(\lambda)$. Similarly, $\rho\delta(1 - \lambda)$ implies $f_\alpha(\rho)\, \delta_\alpha(1 - \gamma)$. Since δ_α is a fuzzy proximity on X_α, it follows that $f_\alpha(\mu)\, \delta_\alpha f_\alpha(\rho)$. Next we see that for every $(\mu, \rho) \in V$ there exist positive integers k, m and $\mu_1, \ldots, \mu_k$, $\rho_1, \ldots, \rho_m$ such that $\mu = \mu_1 \vee \cdots \vee \mu_k$, $\rho = \rho_1 \vee \cdots \vee \rho_m$ and for each pair (i, j), $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant m$, there is an α with $f_\alpha(\mu_i)\, \bar{\delta}_\alpha f_\alpha(\rho_j)$. Let $n = k + m$. Let us call such an n an integer corresponding to (μ, ρ). Of course n is not uniquely determined by (μ, ρ) and as we have shown $n > 2$. Let K be the set of all integers corresponding to members of V and let n be the smallest member of K. Choose a $(\mu, \rho) \in V$ whose one of the corresponding integers is n. We also choose $\mu_1, \ldots, \mu_k$, $\rho_1, \ldots, \rho_m$ in I^X such that $k + m = n$, $\mu = \mu_1 \vee \ldots \vee \mu_k$, $\rho = \rho_1 \vee \ldots \vee \rho_m$ and for each pair (i, j) there exists an α with $f_\alpha(\mu_i)\, \bar{\delta}_\alpha f_\alpha(\rho_j)$. One of the k, m is greater than 1. Suppose, for example, that $k > 1$ and let $\sigma = \mu_1 \vee \cdots \vee \mu_{k-1}$. One of the following should be true:

(A) For every $\gamma \in I^X$, either $\sigma\delta\gamma$ or $\rho\delta(1 - \gamma)$

(B) For every $\gamma \in I^X$, either $\mu_k\delta\gamma$ or $\rho\delta(1 - \gamma)$.

To prove this, suppose that neither (A) nor (B) holds. Then there exist $\gamma_1, \gamma_2 \in I^X$ such that $\sigma\bar{\delta}\gamma_1$, $\mu_k\bar{\delta}\gamma_2$, $\rho\bar{\delta}(1 - \gamma_1)$, $\rho\bar{\delta}(1 - \gamma_2)$. Letting $\gamma = \gamma_1 \wedge \gamma_2$, we have $\gamma\bar{\delta}\mu$ and $\rho\bar{\delta}(1 - \gamma)$ which contradicts our assumption that (μ, ρ) is in V.

Suppose, for example, that (A) holds. Since $\sigma \leqslant \mu$ and $\mu\bar{\delta}\rho$, we have $\sigma\bar{\delta}\rho$. It follows that $(\sigma, \rho) \in V$. Hence $(k - 1) + m = n - 1$ belongs to K which contradicts our choise of n. This contradiction proves that δ is a fuzzy proximity on X.

It is clear that, with respect to δ, each f_α is proximally continuous. Let δ' be another fuzzy proximity on X with respect to which each f_α is a proximity mapping. We will show that δ' is finer than d. In fact, suppose that $\mu\delta'\rho$ and that $\mu = \mu_1 \vee \cdots \vee \mu_n$, $\rho = \rho_1 \vee \cdots \vee \rho_m$. There are i, j such that $\mu_i\delta'\rho_j$. Since each f_α is proximally continuous with respect to δ', it follows that $f_\alpha(\mu_i)\, \delta_\alpha f_\alpha(\rho_j)$ for each α. This proves that $\mu\delta\rho$ and the proof of (a) is complete.

(b) *Let* (Y, δ_1) be a fuzzy proximity space and $f: (Y, \delta_1) \to (X, \delta)$. It is clear that if f is a proximity mapping, then each $h_\alpha = f_\alpha \circ f$ is a proximity mapping. Conversely, suppose that each h_α is proximally continuous and suppose that $\mu\delta_1\rho$. Suppose that $f(\mu) = \mu_1 \vee \cdots \vee \mu_n$, $f(\rho) = \rho_1 \vee \cdots \vee \rho_k$.

Let

$$\mu_0 = f^{-1}(\mu_1) \vee \cdots \vee f^{-1}(\mu_n) \geqslant \mu$$

and

$$\rho_0 = f^{-1}(\rho_1) \vee \cdots \vee f^{-1}(\rho_k) \geqslant \rho.$$

Since $\mu\delta_1\rho$, we have $\mu_0\delta_1\rho_0$. Hence there exist i, j such that $f^{-1}(\mu_i)\, \delta_1 f^{-1}(\rho_j)$. Since

$$h_\alpha(f^{-1}(\mu_i)) \leqslant f_\alpha(\mu_i) \qquad \text{and} \qquad h_\alpha(f^{-1}(\rho_j)) \leqslant f_\alpha(\rho_j),$$

our hypothesis implies that $f_\alpha(\mu_i)\, \delta_\alpha f_\alpha(\rho_j)$ for each $\alpha \in A$. This proves that $f(\mu)\, \delta f(\rho)$ and the Theorem is proved.

5. Subspaces and Products of Fuzzy Proximity Spaces

(5.1) Definition. Let (X, δ) be a fuzzy proximity space and $Y \subset X$. The fuzzy proximity induced by δ on Y (or the subspace proximity) is the coarsest fuzzy proximity δ_Y on Y with respect to which the inclusion map from Y into X is proximally continuous.

We omit the proof of the following easily established Theorem.

(5.2) Theorem. *Let (X, δ) be a fuzzy proximity space, $Y \subset X$ and $\mu, \rho \in I^Y$. The following are equivalent:*

(a) $\mu\delta_Y\rho$

(b) $\mu_X\delta\rho_X$, *where for $\sigma \in I^Y$, σ_X is defined on X by*

$$\sigma_X(z) = 0, \qquad z \notin Y$$
$$\qquad = \sigma(z), \qquad z \in Y.$$

(c) *For all* $\mu_1, \rho_1 \in I^X$ *whose restrictions to* Y *are the* μ, ρ, *respectively, we have* $\mu_1 \delta \rho_1$.

(5.3) DEFINITION. Let $(X_\alpha, \delta_\alpha)_{\alpha \in A}$ be a family of fuzzy proximity spaces and $X = \prod_{\alpha \in A} X_\alpha$. The product fuzzy proximity on X is defined to be the coarsest fuzzy proximity δ on X with respect to which each projection map $p_\alpha \colon X \to X_\alpha$ proximally continuous.

In view of Theorem (4.1), the product fuzzy proximity δ on X exists. Also, it is easy to see that $\tau(\delta)$ is finer than the product of the fuzzy topologies $\tau(\delta_\alpha)$.

6. Fuzzy Proximity Induced by a Proximity

Let δ be a proximity on X. We define a binary relation $\delta_1 = i(\delta)$ on I^X as follows:

For $\mu, \rho \in I^X$, $\mu \delta_1 \rho$ iff there are subsets A, B of X such that $\mu \leqslant \chi_A$, $\rho \leqslant \chi_B$ and $A \delta B$.

(6.1) THEOREM. *Let* (X, δ) *be a proximity space. Then* $i(\delta)$ *is a fuzzy proximity on* X.

Proof. The proof is quite straightforward and we omit it.

(6.2) THEOREM. *Let* (X, δ_0) *be a fuzzy proximity space and consider the following statements:*

(1) *There exists a proximity relation* δ *on* X *such that* $\delta_0 = i(\delta)$.

(2) *If* $\mu \bar{\delta}_0 \rho$, *there are subsets* A, B *of* X *such that* $\mu \leqslant \chi_A$, $\rho \leqslant \chi_B$ *and* $\chi_A \bar{\delta}_0 \chi_B$.

(3) *The relation* $A \delta B$ *iff* $\chi_A \delta_0 \chi_B$ *is a proximity on* X.

Then, (1) *and* (2) *are equivalent and they imply* (3).

Proof. It is clear that (1) implies (2).

(2 $\Rightarrow$ 3). Since the other axioms are easily verified, it suffices to show that δ satisfies (P4). So, assume that A, B are two subsets of X such that $A \bar{\delta} B$. Since $\chi_A \bar{\delta}_0 \chi_B$, there exists $\gamma \in I^X$ such that $\chi_A \bar{\delta}_0 \gamma$ and $\chi_B \bar{\delta}_0 (1 - \gamma)$. By (2), there are subsets E, F, G, L of X such that $\chi_A \leqslant \chi_E$, $\gamma \leqslant \chi_F$, $\chi_B \leqslant \chi_G$, $1 - \gamma \leqslant \chi_L$, $\chi_E \bar{\delta}_0 \chi_F$, $\chi_G \bar{\delta}_0 \chi_L$. From the $\gamma \leqslant \chi_F$ and $1 - \gamma \leqslant \chi_L$ follows that $X - F \subset L$. Now, since $\chi_E \bar{\delta}_0 \chi_F$ and $\chi_A \leqslant \chi_E$, we have $\chi_A \bar{\delta}_0 \chi_F$ and hence $A \bar{\delta} F$. Also, $\chi_G \bar{\delta}_0 \chi_L$, $\chi_B \leqslant \chi_G$ and $X - F \subset L$ imply that

$$\chi_B \bar{\delta}_0 (1 - \chi_F) \qquad \text{and hence} \qquad B \bar{\delta} (X - F).$$

This shows that δ satisfies (P4).

$(2 \Rightarrow 1)$. Define the binary relation δ on the power set of X by $A\delta B$ iff $\chi_A \delta_0 \chi_B$. As we have shown, δ is a proximity on X. We will finish the proof by showing that $\delta_0 = i(\delta)$. To this end, suppose first that $\mu \bar{\delta}_0 \rho$. By (2), there are subsets A, B of X, $\mu \leqslant \chi_A$, $\rho \leqslant \chi_B$ and $\chi_A \bar{\delta}_0 \chi_B$. Thus $\mu \bar{\delta}_0 \rho$ implies that there are subsets A, B of X, $\mu \leqslant \chi_A$, $\rho \leqslant \chi_B$ and $A \bar{\delta} B$ which proves that $\mu i(\bar{\delta}) \rho$. Conversely, assume that $\mu i(\bar{\delta}) \rho$. By the definition of $i(\delta)$, there are subsets A, B of X such that $\mu \leqslant \chi_A$, $\rho \leqslant \chi_B$ and $A \bar{\delta} B$ and hence $\chi_A \bar{\delta}_0 \chi_B$. Thus, if $\mu i(\bar{\delta}) \rho$, we have $\mu \bar{\delta}_0 \rho$ since $\mu \leqslant \chi_A$, $\rho \leqslant \chi_B$ and $\chi_A \bar{\delta}_0 \chi_B$. This shows that $\delta_0 = i(\delta)$ and the proof is complete.

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